MIT OCW GR PSET 1

1 a · Show that the sum of any two orthogonal spacelike vectors is spacelike

"spacelike" $\Rightarrow \vec{A}\cdot\vec{A} > 0 \, , \, \vec{B}\cdot\vec{B} > 0$

"orthogonal" $\Rightarrow \vec{A}\cdot\vec{B} = 0$

$$\vec{A} + \vec{B} \equiv \vec{C}$$

$$\Rightarrow \vec{C}\cdot\vec{C} = \underbrace{\vec{A}\cdot\vec{A}}_{>0} + \underbrace{\vec{B}\cdot\vec{B}}_{>0} + \underbrace{2\vec{A}\cdot\vec{B}}_{0}$$

$\Rightarrow \boxed{\vec{C}\cdot\vec{C} > 0 \text{ and } \vec{A}+\vec{B} \text{ is also spacelike}}$

b · Show that a timelike vector and a null vector cannot be orthogonal

· If $\vec{A}$ timelike, then $\vec{A}\cdot\vec{A} < 0$ $\quad \vec{B} \neq 0$

· If $\vec{B}$ null, then $\vec{B}\cdot\vec{B} = 0$ and [redacted]

· Assume $\vec{A}\cdot\vec{B}$ is orthogonal i.e. $\vec{A}\cdot\vec{B} = 0$

$$-(B^0)^2 + |\vec{B}|^2 = 0$$

$$\Rightarrow (B^0)^2 = |\vec{B}|^2$$

$$-A^0B^0 + A^1B^1 + A^2B^2 + A^3B^3 = 0$$

- Choose a frame where $\vec{A}$ is completely temporal: $\vec{A} = (A^0, 0, 0, 0)$

$\vec{A}\cdot\vec{B}$ is Lorentz invariant so we are free to work in this frame W.L.O.G.

$$\Rightarrow A^0B^0 = 0, \quad \text{Also } (A^0)^2 < 0$$

$$\Rightarrow B^0 = 0$$

$$\Rightarrow (B^0)^2 = |\vec{B}|^2 = 0$$

$$\Rightarrow (B^1)^2 + (B^2)^2 + (B^3)^2 = 0$$

$$\Rightarrow B^1 = B^2 = B^3 = 0$$

$$\Rightarrow \vec{B} = (0, \vec{0})$$

- Which contradicts our definition of a null or "light like" vector $\vec{B}$ and implies

$$\boxed{\vec{A}\cdot\vec{B} \text{ cannot } = 0 \quad \divideontimes}$$

[2] · In some reference frame $\vec{U}$ and $\vec{D}$ have the components:

$$U^\alpha \doteq (1+t^2, t^2, \sqrt{2}t, 0)$$

$$D^\alpha \doteq (x, 5tx, \sqrt{2}t, 0)$$

and the scalar $\rho = x^2 + t^2 - y^2$

$\boxed{a}$ · Show that $\vec{U}$ is suitable as a 4-velocity. Is $\vec{D}$ also?

· a 4-velocity requires that $\vec{U} \cdot \vec{U} = -1$

$$\vec{U} \cdot \vec{U} = -(1+t^2)^2 + t^4 + 2t^2$$
$$= -1 - t^4 - 2t^2 + t^4 + 2t^2$$
$$= \boxed{-1 \quad \checkmark}$$

· What about $\vec{D}$?

$$\vec{D} \cdot \vec{D} = -x^2 + 5^2 t^2 x^2 + 2t^2$$
$$\neq -1 \quad \boxed{\text{So NO}}$$

$\boxed{b}$

· Find the spatial velocity $\vec{V}$ of a particle whose 4-velocity is $\vec{U}$, for arbitrary t:

$$\vec{U} \cdot \vec{U} = -(U^0)^2 + |\vec{V}|^2$$

$$\Rightarrow |\vec{V}|^2 = (U^0)^2 - 1 = 1 + 2t^2 + t^4 - 1$$

$$\Rightarrow |\vec{V}| = \sqrt{2t^2 + t^4} = \boxed{t\sqrt{2+t^2} = |\vec{V}|}$$

$$\boxed{\lim_{t=0} |\vec{V}| = 0, \quad \lim_{t\to\infty} |\vec{V}| \to \infty \approx t^2}$$

c · Find $\partial_\beta U^\alpha$ for all α, β. Show that $U_\alpha \partial_\beta U^\alpha = 0$ via brute force

α	β	$\partial_\beta U^\alpha$
t	t	2t
t	x	0
t	y	0
t	z	0
x	t	2t
x	x	0
x	y	0
x	z	0
y	t	$\sqrt{2}$
y	x	0
y	y	0
y	z	0
z	t	0
z	x	0
z	y	0
z	z	0

· One could cleverly show that:

$$U_\alpha \partial_\beta U^\alpha = \partial_\beta U_\alpha U^\alpha = \partial_\beta (\vec{U} \cdot \vec{U}) = \partial_\beta(-1) = \emptyset \checkmark$$

· But we will use Brute force:

$$U_\alpha \partial_\beta U^\alpha = -U_t \partial_t U^t + U_x \partial_t U^x + U_y \partial_t U^y + U_z \partial_t U^z$$

$$= -(1+t^2)(2t) + (t^2)(2t) + (\sqrt{2}t)(\sqrt{2})$$

$$= \emptyset$$

$$\boxed{\Rightarrow U_\alpha \partial_\beta U^\alpha = \emptyset \checkmark}$$

$\boxed{d}$ · Find $\partial_\alpha D^\alpha$ → represents a set of 4 numbers that we sum together?

$$\partial_\alpha D^\alpha = +\partial_t D^t + \partial_x D^x + \partial_y D^y + \partial_z D^z$$
$$= 0 + 5t + 0 + 0$$

$$\Rightarrow \boxed{\partial_\alpha D^\alpha = 5t}$$

$\boxed{e}$ · Find $\partial_\beta (U^\alpha D^\beta)$ for all α

$$= U^\alpha \partial_\beta D^\beta = \boxed{5t\,U^\alpha}$$ -or equivalently-

$$\boxed{= (5t + 5t^3,\ 5t^4,\ 5\sqrt{2}\,t^2,\ 0)}$$

$\boxed{f}$ · Find $U_\alpha \partial_\beta (U^\alpha D^\beta)$

$$= 5t\, U_\alpha U^\alpha = 5t(\vec{U}\cdot\vec{U}) = \boxed{-5t}$$

· This is similar to $\boxed{d}$ since the expressions are actually equivalent up to a minus sign W :)

$\boxed{g}$ · calculate $\partial_\alpha \rho$ for all α. Calculate $\partial^\alpha \rho$:

$$\partial_\alpha \rho \rightarrow \{\partial_t \rho, \partial_x \rho, \partial_y \rho, \partial_z \rho\}$$

$$\boxed{\partial_\alpha \rho = \{2t, 2x, -2y, 0\}}$$

$\partial^\alpha \rho$ is shorthand for:

$\partial^\alpha \rho = g^{\alpha\beta}\partial_\beta \rho$ where $g^{\alpha\beta}$ is the metric tensor $= \text{diag}(-1,1,1,1)$

· therefore,

$$\partial^\alpha \rho = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2t \\ 2x \\ -2y \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} -2t \\ 2x \\ -2y \\ 0 \end{bmatrix} = \partial^\alpha \rho}$$

h · Find $\nabla_{\vec{U}} \rho$ and $\nabla_{\vec{D}} \rho$:

$\nabla_{\vec{U}} \rho = U^\alpha \partial_\alpha \rho$ $\qquad$ $\nabla_{\vec{D}} \rho = D^\alpha \partial_\alpha \rho$

$$\boxed{2t + 2t^4 + 2xt^2 - 2\sqrt{2}\,yt} \qquad \boxed{2tx + 10x^2t - 2\sqrt{2}\,yt}$$

group like terms

3. Consider a timelike unit 4-vector $\vec{U}$ and the tensor:

$$P_{\alpha\beta} = \eta_{\alpha\beta} + U_\alpha U_\beta$$

a · Show that $V_\perp^\alpha = P^\alpha_{\ \beta} V^\beta$ is orthogonal to $\vec{U}$:

· we know $\vec{U}\cdot\vec{U} < 0$ "timelike" and unit vector

so $\Rightarrow$ $\vec{U}\cdot\vec{U} = -1$

· Let $P^\alpha_{\ \beta} = \eta^\alpha_{\ \beta} + U^\alpha U_\beta$

$$\Rightarrow P^\alpha_{\ \beta} V^\beta = \eta^\alpha_{\ \beta} V^\beta + U^\alpha U_\beta V^\beta \equiv V_\perp^\alpha$$

$$\Rightarrow U_\alpha V_\perp^\alpha = \underbrace{U_\alpha \eta^\alpha_{\ \beta} V^\beta}_{\vec{U}\cdot\vec{V}} + \underbrace{U_\alpha U^\alpha}_{-1} \underbrace{U_\beta V^\beta}_{\vec{U}\cdot\vec{V}}$$

$$\Rightarrow U_\alpha V_\perp^\alpha = \vec{U}\cdot\vec{V}_\perp = \vec{U}\cdot\vec{V} - \vec{U}\cdot\vec{V} = 0$$

$$\Rightarrow \boxed{\vec{U}\cdot\vec{V}_\perp = 0 \ \checkmark}$$

b · Show that $V_\perp^\alpha = P^\alpha_\beta V^\beta$ is unaffected by further projections:

· i.e. Show that $V_{\perp\perp}^\alpha \equiv P^\alpha_\beta V_\perp^\beta = V_\perp^\alpha$

$$P^\alpha_\beta V_\perp^\beta = \eta^\alpha_\beta V_\perp^\beta + U^\alpha \underbrace{U_\beta V_\perp^\beta}_{\vec{U}\cdot\vec{V}_\perp = 0} \equiv V_{\perp\perp}^\alpha$$

$$= \eta^\alpha_\beta V_\perp^\beta = V_\perp^\alpha$$

$$\Rightarrow \boxed{V_{\perp\perp}{}^\alpha = V_\perp{}^\alpha \quad \checkmark}$$

c · Show that $P_{\alpha\beta} V_\perp^\alpha W_\perp^\beta = \vec{V}_\perp \cdot \vec{W}_\perp$

$$\text{RHS} = \vec{V}_\perp \cdot \vec{W}_\perp = V_\perp^\alpha W_{\perp\alpha}$$

$$= P_\beta{}^\alpha V^\beta P_\alpha{}^\gamma W_\gamma = V^\beta W_\gamma (\eta^\alpha_\beta + U^\alpha U_\beta)(\eta^\gamma_\alpha + U^\gamma U_\alpha)$$

$$= V^\beta W_\gamma \delta^\gamma_\beta + V^\beta W_\gamma U^\alpha U_\alpha U_\beta U^\gamma$$

$$+ V^\beta W_\gamma U^\alpha U_\beta \eta^\gamma_\alpha + V^\beta W_\gamma \eta^\alpha_\beta U^\gamma U_\alpha$$

$$= V^\beta W_\beta - (\vec{U}\cdot\vec{V})(\vec{V}\cdot\vec{W}) + 2(\vec{U}\cdot\vec{V})(\vec{U}\cdot\vec{W})$$

$$= (\vec{V}\cdot\vec{W}) + (\vec{U}\cdot\vec{V})(\vec{U}\cdot\vec{W}) \quad \checkmark$$

$$\text{LHS} = P_{\alpha\beta} V_\perp^{\alpha} W_\perp^{\beta}$$

$$= (\eta_{\alpha\beta} + U_\alpha U_\beta)(P^\alpha_\gamma V^\gamma)(P^\beta_\chi W^\chi)$$

$$= (\eta_{\alpha\beta} + U_\alpha U_\beta)(\eta^\alpha_\gamma + U^\alpha U_\gamma)(\eta^\beta_\chi + U^\beta U_\chi) V^\gamma V^\chi$$

$$= \left(\eta_{\alpha\beta}\eta^\alpha_\gamma + U_\alpha U_\beta \eta^\alpha_\gamma + \eta_{\alpha\beta} U^\alpha U_\gamma + U_\alpha U_\beta U^\alpha U_\gamma\right) \cdot (\eta^\beta_\chi + U^\beta U_\chi) V^\gamma W^\chi$$

$$= \eta_{\alpha\beta}\eta^\alpha_\gamma \eta^\beta_\chi V^\gamma W^\chi + U_\alpha U_\beta \eta^\alpha_\gamma \eta^\beta_\chi V^\gamma W^\chi$$
$$+ \eta_{\alpha\beta}\eta^\beta_\chi U^\alpha U_\gamma V^\gamma W^\chi + U_\alpha U_\beta U^\alpha U_\gamma \eta^\beta_\chi V^\gamma W^\chi$$
$$+ \eta_{\alpha\beta}\eta^\alpha_\gamma U^\beta U_\chi V^\gamma W^\chi + U_\alpha U_\beta U^\beta U_\chi \eta^\alpha_\gamma V^\gamma W^\chi$$
$$+ \eta_{\alpha\beta} U^\alpha U_\gamma U^\beta U_\chi V^\gamma W^\chi + U_\alpha U_\beta U^\alpha U_\gamma U^\beta U_\chi V^\gamma W^\chi$$

$= (\vec{V}\cdot\vec{W}) + (\vec{U}\cdot\vec{V})(\vec{U}\cdot\vec{W}) +$ [redacted]

[redacted] $\cancel{(\vec{U}\cdot\vec{V})(\vec{U}\cdot\vec{W})} - \cancel{(\vec{U}\cdot\vec{V})(\vec{U}\cdot\vec{W})}$

$+ \cancel{(\vec{U}\cdot\vec{V})(\vec{U}\cdot\vec{W})} - \cancel{(\vec{U}\cdot\vec{V})(\vec{U}\cdot\vec{W})} - \cancel{(\vec{U}\cdot\vec{V})(\vec{V}\cdot\vec{W})}$

$+ \cancel{(\vec{U}\cdot\vec{V})(\vec{U}\cdot\vec{W})} = (\vec{V}\cdot\vec{W}) + (\vec{U}\cdot\vec{V})(\vec{U}\cdot\vec{W})$

$$\Rightarrow \text{LHS} = \text{RHS} \Rightarrow \boxed{P_{\alpha\beta} V_\perp^\alpha W_\perp^\beta = \vec{V}_\perp \cdot W_\perp} \quad ✓$$

$\boxed{d}$ · Show that for an arbitrary non-null vector $\vec{q}$, the projection tensor is given by:

$$P_{\alpha\beta}(q^\alpha) = \eta_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^\gamma q_\gamma}$$

· If this is the case, then $q_\perp \equiv \eta_{\alpha\beta} - \dfrac{q_\alpha q_\beta}{q^\gamma q_\gamma}$ and q^α should be orthogonal:

$$q_{\perp\alpha} q^\alpha = \eta_{\alpha\beta} q^\alpha - \frac{q_\alpha q^\alpha q_\beta}{q^\gamma q_\gamma}$$

$$= q_\beta - q_\beta = 0 \quad ✓$$

· It doesn't make sense to have a projection tensor for null vectors since every vector is orthogonal to a null vector... ✓

$\boxed{4}$ · Let $\Lambda_B(\bar{v})$ be a Lorentz boost associated with 3 velocity $\bar{v}$. Consider

$$\Lambda_{tot} \equiv \Lambda_B(\bar{v}_1)\Lambda_B(\bar{v}_2)\Lambda_B(-\bar{v}_1)\Lambda_B(-\bar{v}_2)$$

where $\bar{v}_1 \cdot \bar{v}_2 = 0$ and $v_1, v_2 \ll 1$. Show that Λ_{tot} is a rotation. What is the axis + angle.

· Intuitively it makes sense that the composition of these 4 boosts should generally get you back to the same frame, but since they don't commute, there is some mixing going on that causes these rotations.

· Take the case:

$$\Lambda_{tot} = \Lambda_B(v_x)\Lambda_B(v_y)\Lambda_B(-v_x)\Lambda_B(-v_y)$$

where $v_x \perp v_y$

$$\Lambda_B(v_x) = \begin{bmatrix} \gamma_1 & -\gamma_1\beta_1 & 0 & 0 \\ -\gamma_1\beta_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and $\rightsquigarrow$

$$\Lambda_B(v_y) = \begin{bmatrix} \gamma_2 & 0 & -\gamma_2\beta_2 & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma_2\beta_2 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and for $\Lambda_B(-v_x)$ and $\Lambda_B(-v_y)$ they are similar but all the entries are positive

$$\Rightarrow \Lambda_B(-v_x)\Lambda_B(-v_y) = \begin{bmatrix} +\gamma_1 & +\gamma_1\beta_1 & 0 & 0 \\ +\gamma_1\beta_1 & +\gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_2 & 0 & \gamma_2\beta_2 & 0 \\ 0 & 1 & 0 & 0 \\ \gamma_2\beta_2 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \gamma_1\gamma_2 & \gamma_1\beta_1 & \gamma_1\gamma_2\beta_2 & 0 \\ \gamma_1\gamma_2\beta_1 & \gamma_1 & \gamma_1\gamma_2\beta_1\beta_2 & 0 \\ \gamma_2\beta_2 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

• Also

$$\Lambda_B(v_x)\Lambda_B(v_y) = \begin{bmatrix} \gamma_1 & -\gamma_1\beta_1 & 0 & 0 \\ -\gamma_1\beta_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_2 & 0 & -\gamma_2\beta_2 & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma_2\beta_2 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \gamma_1\gamma_2 & -\gamma_1\beta_1 & -\gamma_1\gamma_2\beta_2 & 0 \\ -\gamma_1\gamma_2\beta_1 & \gamma_1 & \gamma_1\gamma_2\beta_1\beta_2 & 0 \\ -\gamma_2\beta_2 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Lambda_{tot} = \Lambda(v_x)\Lambda(v_y)\Lambda(-v_x)\Lambda(-v_y)$$

$$= \begin{bmatrix} \gamma_1\gamma_2 & -\gamma_1\beta_1 & -\gamma_1\gamma_2\beta_2 & 0 \\ -\gamma_1\gamma_2\beta_1 & \gamma_1 & \gamma_1\gamma_2\beta_1\beta_2 & 0 \\ -\gamma_2\beta_2 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_1\gamma_2 & \gamma_1\beta_1 & \gamma_1\gamma_2\beta_2 & 0 \\ \gamma_1\gamma_2\beta_1 & \gamma_1 & \gamma_1\gamma_2\beta_1\beta_2 & 0 \\ \gamma_2\beta_2 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \gamma_1^2\gamma_2^2 - \gamma_1^2\gamma_2\beta_1^2 - \gamma_1\gamma_2^2\beta_2^2 & \gamma_1^2\gamma_2\beta_1 - \gamma_1^2\beta_1 & \gamma_1^2\gamma_2^2\beta_2 - \gamma_1^2\gamma_2\beta_1^2\beta_2 - \gamma_1\gamma_2^2\beta_2 & 0 \\ -\gamma_1^2\gamma_2^2\beta_1 + \gamma_1^2\gamma_2\beta_1 + \gamma_1\gamma_2^2\beta_1\beta_2^2 & -\gamma_1^2\gamma_2\beta_1^2 + \gamma_1^2 & -\gamma_1^2\gamma_2^2\beta_2\beta_1 + \gamma_1^2\gamma_2\beta_1\beta_2 + \gamma_1\gamma_2^2\beta_1\beta_2 & 0 \\ -\gamma_1\gamma_2^2\beta_2 + \gamma_2^2\beta_2 & -\gamma_1\gamma_2\beta_1\beta_2 & \gamma_2^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- This huge annoying 4x4 matrix should in theory represent a rotation around z. Therefore, we compare the circled 3x3 matrix to a Rotation matrix.

$$\begin{bmatrix} -\gamma_1^2\gamma_2\beta_1^2 + \gamma_1^2 & -\gamma_1^2\gamma_2^2\beta_1\beta_2 + \gamma_1^2\gamma_2\beta_1\beta_2 + \gamma_1\gamma_2^2\beta_1\beta_2 & 0 \\ -\gamma_1\gamma_2\beta_1\beta_2 & \gamma_2^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\stackrel{?}{=} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightsquigarrow$$

• That would imply:

$$\sin\theta = -\gamma_1\gamma_2\beta_1\beta_2$$

$$\theta = \frac{-v_1 v_2}{\sqrt{(1-v_1^2)(1-v_2^2)}}$$

units with $c = 1$

denominator ≈ 1 since $v_1 \ll 1$ and $v_2 \ll 1$

$\Rightarrow |\theta| \approx v_1 v_2$ about the z axis.

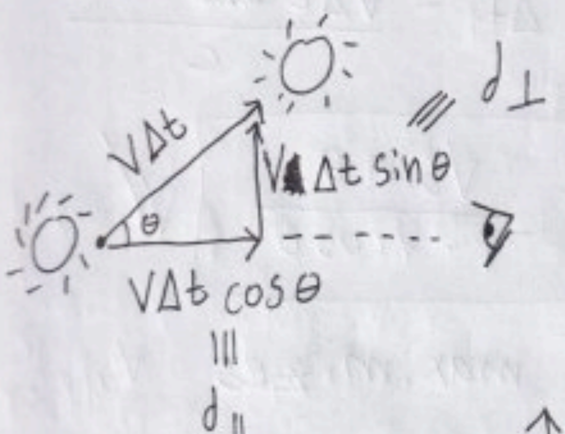
Back in units with $c = c$, we get

$$\boxed{|\theta| \approx \frac{v_1 v_2}{c^2} \quad \text{about } z\text{-axis}}$$

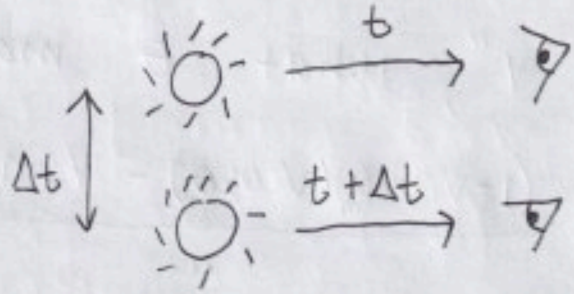
5 . A quasar is moving towards you and up at an angle θ. The apparent upwards velocity is V_{app}:

a . Find the expression for V_{app} in terms of θ and the true velocity "V":

· Let's draw a diagram:

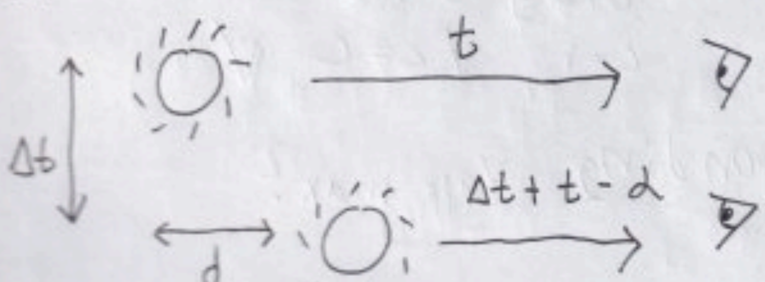


· Also take an aside & start to think about light discretely. Imagine a light bulb releasing photons at intervals of Δt:



$\Delta t_{observed} =$
$(\Delta t + t) - t$
$= \Delta t$

· If the light bulb is stationary, then you also perceive the time between flashes as Δt as above. However, if the bulb is moving towards you:



$\Delta t_{obs} = (\Delta t + t - \frac{d}{c}) - t$
$= \Delta t - d/c$

define $\alpha \equiv d/c$

· Then $\Delta t_{obs} = \Delta t - d/c \neq \Delta t$
i.e. you see a different time between emission of photons compared to the proper time of the bulb.

· We can say that the perceived upwards velocity of the quasar is:

$$V_{app} = \frac{d_\perp}{\Delta t_{obs}} \; ; \quad \Delta t_{obs} = \Delta t - d_{\parallel}/c = \Delta t - \frac{V \Delta t \cos\theta}{c}$$

$$\Rightarrow V_{app} = \frac{V \Delta t \sin\theta}{\Delta t - V \Delta t \cos\theta} \Rightarrow \boxed{V_{app} = \frac{V \sin\theta}{1 - V\cos\theta}}$$

[b] · For a given "V", what θ maximizes V_{app}?

$$0 = \left.\frac{\partial V_{app}}{\partial \theta}\right|_{\theta_{max}} = \frac{(1 - V\cos\theta_m)(V\cos\theta_m) - V\sin\theta_m(V\sin\theta_m)}{\cdots}$$

$$\Rightarrow V\cos\theta_{max} - V^2(\sin^2\theta_{max} + \cos^2\theta_{max}) = 0$$

$$\Rightarrow \cos\theta_{max} = V \Rightarrow \boxed{\theta_{max} = \cos^{-1}(V)}$$

okay since in units with $c = 1$, $V <= c$

· What is the corresponding $v_{app,max}$?

bounded to <= 1

$$v_{am} = \frac{v \sin(\cos^{-1}(v))}{\underbrace{1 - v^2}}, \quad v < 1$$

really small if $v \ll 1$

$\Rightarrow$ $\boxed{\text{possible for } v_{am} > 1 \quad (\text{in units with } c = 1)}$

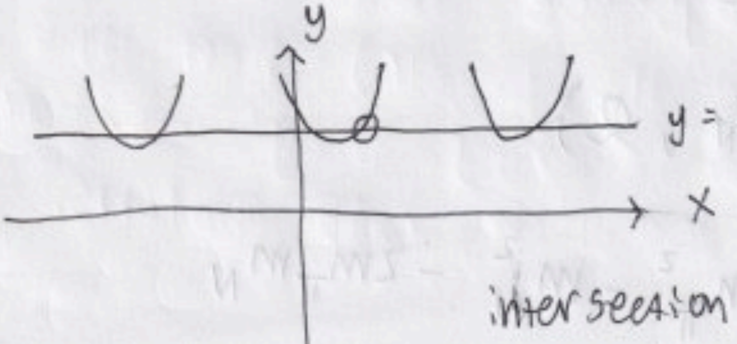
- Special Relativity is not violated in this case because this is just the apparent motion / information is not actually being transmitted at $> c$ values...

$\boxed{c}$ · For $v_{app} \approx 10c$, what is the largest possible value of θ?

i.e. $10 = \dfrac{v \sin\theta}{1 - v\cos\theta}$ · plot this as $\theta \to x$ and $v \to y$ and ensure $y < 1$:

$\Rightarrow \quad 10 - 10y\cos x = y \sin x$

$\Rightarrow \quad \dfrac{10}{\sin x + 10\cos x} = y \quad ; \quad y <= 1$

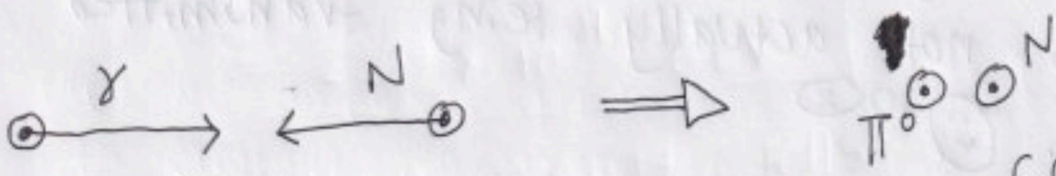


radians

intersection at $(0.19934, 1)$

$\Rightarrow\ 0.199 \times \frac{180°}{\pi} \approx$ $\boxed{\text{11.4° is max possible } \theta \text{ for this configuration}}$

$\boxed{6}$ $\boxed{a}$ · Calculate the threshold energy of a nucleon N for it to undergo the reaction $\gamma + N \rightarrow N + \pi^0$ where γ is a CMB photon w/ energy kT where $T = 2.73$ kelvin. Assume the collision is head on + take $M_N = 938$ MeV and $M_\pi = 135$ MeV:



Lab Frame, Before Collision

COM Frame, After collision (0 relative velocity s.t. min energy required)

· $\vec{p}^{\nu}$ is a conserved quantity (the momentum 4-vector) and $(p^{\nu})^2$ {the square of 4-mom between frames}. is a Lorentz invariant. Therefore, to solve relativistic kinematics problems like this we will compare $(p^{\nu})^2_{\text{initial}} = (p^{\nu})^2_{\text{final}}$

· In the COM frame after collision:

$$\vec{P}_{tot} = (m_\pi + m_N, 0)$$

$$\Rightarrow (p^{\nu})^2_f = -m_\pi^2 - m_N^2 - 2m_\pi m_N$$

· In the lab frame before the collision,

$$\vec{P}_{tot} = \vec{P}_\gamma + \vec{P}_N$$ [redacted]

$$\Rightarrow (p^\mu)^2_i = p_\gamma^2 + p_N^2 + 2\vec{p}_\gamma \cdot \vec{p}_N$$

Lorentz Invariants

· <u>In lab frame</u>; $\vec{P}_\gamma = (\varepsilon, -\vec{\varepsilon})$

$\varepsilon = kT$

· Since $E^2 = p^2 + m^2$ ← zero rest mass for photon

$$\Rightarrow |\vec{P}|_{\substack{\text{spatial} \\ \text{photon}}} = \varepsilon$$

$$\Rightarrow p_\gamma^2 = -\varepsilon^2 + \varepsilon^2 = 0$$

· <u>Also in lab frame</u>; $\vec{P}_N \approx (E_N, \vec{E}_N)$

since we assume $|\vec{P}_{N,\text{spatial}}|$ [redacted] $\gg m_N$

$$\Rightarrow 2\vec{P}_\gamma \cdot \vec{P}_N \approx 2(-\varepsilon E_N - \varepsilon E_N) = -4kTE_N$$

· <u>In rest frame of Nucleon</u>; $\vec{P}_N = (m_N, \mathbb{0})$

$$\Rightarrow p_N^2 = -m_N^2$$

• Putting this all together yields:

$$-\cancel{m_N^2} - m_\pi^2 - 2m_\pi m_N = -\cancel{m_N^2} - 4kTE_N$$

$$\Rightarrow \boxed{E_N = \frac{m_\pi^2 + 2m_\pi m_N}{4kT}}$$

$$\Rightarrow \boxed{\begin{aligned} E_N &\approx 2.89 \times 10^{20} \text{ eV} \\ &= 2.89 \times 10^{11} \text{ GeV} \end{aligned}}$$

b • The above result tells us that it is very likely any nucleons above this energy would collide with CMB photons + be "destroyed". It would then seem unlikely to find any freely traveling nucleons above this energy. This is called the Griesen-Zatsepin-Kuzmin GZK cutoff.

c • Many examples of cosmic rays above the GZK cutoff have been found. These are puzzling, but if these cosmic rays are made of nucleons heavier than a proton (like He, Li, etc) then this would raise the threshold energy they could achieve before they collide with CMB photons.